INTEGRABLE SYSTEMS AND RANK ONE CONDITIONS FOR RECTANGULAR MATRICES

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ABSTRACT. We provide a determinantal formula for tau-functions of the KP hierarchy in terms of *rectangular*, constant matrices A, B and C satisfying a rank one condition. This result is shown to generalize and unify many previous results of different authors on constructions of tau-functions for differential and difference integrable systems from *square* matrices satisfying rank one conditions. In particular, it contains as explicit special cases the formula of Wilson for tau-functions of rational KP solutions in terms of Calogero-Moser Lax matrices as well as our previous formula for KP tau functions in terms of almost-intertwining matrices.

1. INTRODUCTION

In many recent papers by different authors, determinantal formulas have been used to transform constant *square* matrices satisfying a rank one condition into tau-functions for integrable systems. In particular, we recall the result of G. Wilson [19] that $n \times n$ matrices X and Z satisfying the "almost-canonically conjugate" condition

$$(1) \qquad \operatorname{rank}\left([X, Z] + I\right) = 1$$

produce tau-functions for rational solutions to the KP hierarchy by the formula

$$(2) \qquad \tau(\vec{t}) = \det\left(X + \sum_{i=1}^{\infty} it_i Z^{i-1}\right).$$

(See also [2, 3, 5, 14] .)

The previous result can be interpreted as a relationship between the KP hierarchy and the Calogero-Moser particle system and is therefore similar to the relationship between KdV solitons and the Ruijsenaars-Schneider particle system [4, 15]. In that case one finds that the Lax matrices X and Z for this particle system satisfy the rank one condition $\operatorname{rank}(XZ + ZX) = 1$ and that the formula

$$(3) \qquad \tau(\vec{t}) = \det\left(\exp(\sum_{i=0}^{\infty} t_{2i+1} Z^{2i+1}) X \exp(\sum_{i=0}^{\infty} t_{2i+1} Z^{2i+1}) + I\right)$$

gives a tau-function for a multi-soliton solution of the KdV hierarchy.

Both soliton and rational solutions were produced by a formula in our previous paper [11] in which it was shown that square matrices X, Y and Z satisfying the "almost-intertwining" condition

$$(4) \qquad \operatorname{rank}(XZ - YX) = 1$$

produce KP tau-functions through the formula

$$(5) \qquad \tau(\vec{t}) = \det\left(X\exp(\sum_{i=1}^{\infty} t_i Z^i) + \exp(\sum_{i=1}^{\infty} t_i Y^i)\right).$$

Previous results have also demonstrated the usefulness of matrices satisfying these rank one equations in constructing solutions to difference equations. In particular, it was shown in [10] that for matrices X and Z satisfying (1), the eigenvalues x_i^m of the matrix

$$(6) \qquad \mathbf{X}(m) = -\eta X \cdot (\lambda_1 - Z) - m\eta(\lambda_2 - Z)^{-1} \cdot (\lambda_1 - Z)$$

(where η and λ_i are arbitrarily selected constants) satisfy the rational nested Bethe Ansatz equations

$$(7) \qquad \prod_{k=1}^{n} \frac{(x_j^m - x_k^{m-1})(x_j^m - x_k^m + \eta)(x_j^m - x_k^{m+1} - \eta)}{(x_j^m - x_k^{m-1} + \eta)(x_j^m - x_k^m - \eta)(x_j^m - x_k^{m+1})} = -1 \qquad \forall 1 \le j \le n.$$

Moreover, we announced at NEEDS 2001 (an immediate consequence of [11]) that the formula

$$(8)$$
$$\tau_l^{m,n} = \det\left[X(c_1 - Z)^l(c_2 - Z)^m(c_3 - Z)^n + (c_1 - Y)^l(c_2 - Y)^m(c_3 - Y)^n W\right]$$

gives a solution to the Hirota Bilinear Difference equation

$$(9) \quad (c_2 - c_3)\tau_{l+1}^{m,n}\tau_l^{m+1,n+1} - (c_1 - c_3)\tau_l^{m+1,n}\tau_{l+1}^{m,n+1} + (c_1 - c_2)\tau_l^{m,n+1}\tau_{l+1}^{m+1,n} = 0$$

when X, Y and Z satisfy (4) (for arbitrary $n \times n$ matrix W).

In addition, two formulas have recently been published for Baker-Akhiezer functions of the q-KP hierarchy in terms of matrices satisfying q-variants of these "almost" operator identities. In particular, the formula

$$(10) \qquad \psi(x,z) = \frac{\det(xzI + xqZ - zX - XZ - I)}{\det(xI - X)\det(zI + qZ)} e^{xz}$$

in the case rank $(XZ - qZX + I) = 1)$ is found in [6]. Similarly, in [13] one finds the formula

$$(11) \qquad \psi(x,z) = \frac{\det(qZX - xZ - zX + zxI)}{\det(ZX - xZ - zX + zxI)} e_q^{xz}$$

for the case rank $(XZ - qZX) = 1$. In both cases, the interest in these particular Baker-Akhiezer functions lies in their bispectrality (cf. [9]).

Besides the obvious similarities, some explicit connections have been drawn between these results. For example, the formula (6) is easily derived from (2) (although this is not the way it was derived in [10]), formula (5) is able to construct the rational solutions like (2) as well as soliton solutions, and (3) can be viewed as a special case of (5). The formulas concerning the q-KP hierarchy should be able to be related to the others by the correspondence in [6] (cf. [1]) between tau-functions and wave functions of the q-KP and KP hierarchies. However, what is clearly lacking is a general framework in which all of these different formulas appear as special cases. It is our goal in this paper to provide such a framework by generalizing the formulas to the case of *rectangular* matrices.

2. Main Results

Let A, C be full rank $n \times N$ matrices and let B be an $N \times N$ matrix for $N > n$. For convenience, we will assume the non-degeneracy condition $\det\left[AC^{\top}\right] \neq 0$ is satisfied.

Let $L = \operatorname{row} A$ be the subspace in $\mathbb{C}^N$ spanned by the rows of A and $L^{\perp}$ be its orthogonal complement with respect to the standard bilinear form $\langle x, y \rangle = \sum_{k=1}^{N} x_i y_i$. Fix a basis $u_1, \ldots, u_{N-n}$ of $L^{\perp}$ and define U to be the $(N-n) \times N$ matrix with rows $u_1, \ldots, u_{N-n}$. As in [11], let $g(x) = \sum_{i=1}^{\infty} t_i x^i$ be a power series in x with coefficients that depend on the time variables $\vec{t} = (t_1, t_2, \ldots)$ of the KP hierarchy.

Theorem 2.1. *If*

$$(12) \qquad\qquad \operatorname{rank}(ABU^{\top}) \leq 1,$$

then

$$(13) \qquad\qquad \tau_l^{m,n} = \det\left[A(c_1 I - B)^l (c_2 I - B)^m (c_3 I - B)^n C^{\top}\right]$$

is a solution to the Hirota Bilinear Difference Equation (9) *and*

$$(14) \qquad\qquad \tau(\vec{t}) = \det\left[Ae^{g(B)}C^{\top}\right]$$

is a tau-function of the KP hierarchy.

Proof. It is known (see, e.g. [12, 20]) that to prove that $\tau(\vec{t})$ is a KP tau-function it is sufficient to prove that $\tau_l^{m,n} := \tau(\vec{t} - l[c_1^{-1}] - m[c_2^{-1}] - n[c_3^{-1}])$ solves the HBDE for all values of the parameters.[1] Hence, our method will be to prove the second claim above by proving the first. Moreover, it is important to note that it is suficient to prove that HBDE is satisfied when all of the discrete "times" $l, m, n = 0$. This is because $\tau_l^{m,n} = \hat{\tau}_0^{0,0}$ if $\hat{\tau}$ is the tau-function corresponding to the same choice of A and B but with a different C (multiplied on the right by the transpose of $(c_1 I - B)^l \cdots (c_3 - B)^n$). Since the condition in the claim depends only on a property of A and B, it is therefore sufficient to consider the restricted version of the equation

$$(c_2 - c_3)\tau(\vec{t} - [c_1^{-1}])\tau(\vec{t} - [c_2]^{-1} - [c_3^{-1}]) - (c_1 - c_3)\tau(\vec{t} - [c_2^{-1}])\tau(\vec{t} - [c_1^{-1}] - [c_3^{-1}])$$
$$+(c_1 - c_2)\tau(\vec{t} - [c_3^{-1}])\tau(\vec{t} - [c_1^{-1}] - [c_2^{-1}]) \;\; = \;\; 0.$$

We will reduce this equation to the following identity, proved in [11]:

$$(15) \qquad h_1(c_1)h_2(c_2, c_3) - h_1(c_2)h_2(c_1, c_3) + h_1(c_3)h_2(c_1, c_2) \equiv 0,$$

where $h_1(c_1) = \det[c_1 - P]$, $h_2(c_1, c_2) = \det[(c_1 - P)(c_2 - P) + Q]$ with P, Q $n \times n$ matrices and $\operatorname{rank}(Q) \leq 1$.

First, let $V^{\top}$ be any right inverse of A and define $G = [V^{\top} \; U^{\top}]$, $\hat{B} = G^{-1}BG$, $\hat{C}^{\top} = G^{-1}C^{\top}$ and $M = M(\vec{t}) = [I_n \; 0]e^{g(\hat{B})}\hat{C}^{\top}$. Then $AG = [I_n \; 0]$ and $\tau(\vec{t}) = \det\left[M(\vec{t})\right]$. Note also that $G^{-1} = \begin{bmatrix} A \\ * \end{bmatrix}$.

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[1] As usual, the Miwa shift $\vec{t} + c[z]$ is defined by

$$\vec{t} + c[z] = (t_1 + cz, t_2 + \frac{cz^2}{2}, t_3 + \frac{cz^3}{3}, \ldots).$$

The Miwa shift $\tau(\vec{t} - [c^{-1}])$ can be computed as follows:

$$\tau(\vec{t}-[c^{-1}]) = \det\left[[I_n 0]e^{g(\hat{B})}e^{\ln(I_N - c^{-1}\hat{B})}\hat{C}^\top\right] = c^{-n}\tau(\vec{t})\det\left[c - [I_n 0]\hat{B}e^{g(\hat{B})}\hat{C}^\top M(\vec{t})^{-1}\right]$$

and, similarly,

$$\tau(\vec{t} - [c_1^{-1}] - [c_2^{-1}]) = c_1^{-n}c_2^{-n}\tau(\vec{t})\det\left[[I_n\ 0](c_1 - \hat{B})(c_2 - \hat{B})e^{g(\hat{B})}\hat{C}^\top M(\vec{t})^{-1}\right]$$

Then it is not hard to check that a left hand side of the bilinear difference Hirota equation is proportional to the left hand side of (15), if one defines $P = [I_n\ 0]\hat{B}e^{g(\hat{B})}\hat{C}^\top M^{-1}$ and $Q = [I_n\ 0](\hat{B})^2e^{g(\hat{B})}\hat{C}^\top M^{-1} - P^2$. Hence, it suffices to show that $\text{rank}(Q) \leq 1$. Rewrite Q as

$$Q = [I_n\ 0]\hat{B}\left(I - e^{g(\hat{B})}\hat{C}^\top M^{-1}[I_n\ 0]\right)\hat{B}e^{g(\hat{B})}\hat{C}^\top M^{-1}$$

and notice that $e^{g(\hat{B})}\hat{C}^\top = \begin{bmatrix} M \\ * \end{bmatrix}$. Therefore,

$$I - e^{g(\hat{B})}\hat{C}^\top M^{-1}[I_n\ 0] = \begin{bmatrix} 0 & 0 \\ * & I_{N-n} \end{bmatrix},$$

which means that

$$[I_n\ 0]\hat{B}\left(I - e^{g(\hat{B})}\hat{C}^\top M^{-1}[I_n\ 0]\right) = \left([I_n\ 0]\hat{B}\begin{bmatrix} 0 \\ I_{N-n} \end{bmatrix}\right)[*\ I_{N-n}].$$

But the factor in parentheses is equal to $ABU^\top$ and, by our assumption, has a rank less or equal to 1. Therefore, the same is true for Q, which finishes the proof. $\square$

Remark: It is worth emphasizing that the rank-one condition (12) does not depend on a choice of bases in subspaces L and $L^\perp$ that correspond to the matrix A.

Since we have constructed a solution to the KP hierarchy, one may be interested in knowing where in the Sato-Segal-Wilson grassmannian [16, 17] the corresponding solutions lie. This can be resolved by studying the associated Baker-Akhiezer function, $\psi(x, z)$. In particular, one knows that the point $W \in Gr$ is the subspace spanned by ψ and its x-derivatives evaluated at $x = 0$

$$W = \langle \psi(0, z), \psi_x(0, z), \psi_{xx}(0, z), \ldots \rangle.$$

We will show below that there exist polynomials $p(z)$ and $q(z)$ such that $p(z)H_+ \subset W \subset q^{-1}(z)H_+$, which shows by definition that W is in the subgrassmannian Gr^{rat} [7, 18].

Theorem 2.2. *The stationary Baker-Akhiezer function corresponding to the tau-function given in (14) is*

$$(16) \qquad \psi(x, z) = \frac{\det(Ae^{xB}(zI - B)C)}{z^N \det(Ae^{xB}C)}e^{xz}.$$

As a consequence, we are able to determine that this solution corresponds to a point in the subgrassmannian Gr^{rat} of rank one KP solutions with rational spectral curves.

Proof. The formula for $\psi(\vec{t}, z)$ follows from the well known formula for the time-dependent wave function [17]:

$$\psi(\vec{t}, z) = \frac{\tau(\vec{t} - [z^{-1}])}{\tau(\vec{t})}e^{g(z)}$$

which simplifies when evaluated at $\vec{t} = (x, 0, 0, 0, \ldots)$ to the formula above since the coefficients of $-[x]$ are precisely the coefficients in the power series expansion of $\log(1 - x)$.

Another way to identify the subspace W is by its duality with the dual Baker-Akhiezer function ψ^* which by similar arguments to above can be shown to have the property that $p(z)\psi^*$ is nonsingular in z for $p(z) = \det(zI - B)$. Hence, the innerproduct of any polynomial in $p(z)H_+$ with ψ^* (computed as the path integral of the product around S^1) is zero. Consequently, $p(z)H_+ \subset W$ and we see that W is in Gr^{rat}. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It has frequently been found to be useful in the case of solutions corresponding to rational spectral curves [7, 8, 17, 18] to identify a solution instead by the finite dimensional space of finitely supported distributions in z that annihilate the Baker-Akhiezer function. A consequence of the role of the characteristic polynomial $p(z)$ in the proof above is that the finitely supported distributions in z which annihilate $z^N \psi(x, z)$ are supported at the eigenvalues of B with highest derivative taken bounded by the algebraic multiplicity of the eigenvalue.

3. Special Cases

As a corollary to the main theorem above, we can determine the following generalization of our theorem from [11]:

Theorem 3.1. *Let X be an $n \times (N - n)$ matrix, Y be an $n \times n$ matrix and Z be an $(N - n) \times (N - n)$ matrix such that*

$$\mathrm{rank}\,(XZ - YX) \leq 1$$

then (14) *is a tau-function of the KP hierarchy where $A = [X \; I_n]$, B is the block diagonal matrix $B = \mathrm{diag}[Z, Y]$ and C is an arbitrary full rank $n \times (N - n)$ matrix.*

To see that this is so, note that U in (12) can be chosen to be $[-I_n \; X^\top]$. One then finds that condition (12) reads $\mathrm{rank}\,(XZ - YX) \leq 1$. This generalizes (4), since there it was assumed that $N = 2n$ and X is a square matrix. In the latter case, one can define C to be $C = [I_n \; I_n]$, which transforms (14) into a tau-function $\tau(\vec{t}) = \det\left[Xe^{g(Z)} + e^{g(Y)}\right]$ that coincides with (5).

In fact, combining the discussion of the previous paragraph with the observation that the distributions annihilating the Baker-Akhiezer function are supported at the eigenvalues of B with order bounded by the algebraic multiplicity confirms the conjecture in [11] that the same was true for the eigenvalues of Y and Z.

Moreover, one can also rederive Wilson's formula (2) and the "almost-canonically conjugate" rank one condition (1) as a special case of our main result. Consider the case when $N = 2n$, A, C and U are defined as in above but B is chosen in the form $B = \begin{bmatrix} Z & 0 \\ I_n & Z \end{bmatrix}$. Then $ABU^\top = -\left([X, Z] + I_n\right)$ and thus, (12) coincides with (1). Moreover, in this case $e^{g(B)} = \begin{bmatrix} e^{g(Z)} & 0 \\ g'(Z)e^{g(Z)} & e^{g(Z)} \end{bmatrix}$, and the tau-function (14) becomes $\tau(\vec{t}) = \det[e^{g(Z)}] \det[X + g'(Z)]$, which is gauge equivalent to the one in (2).

In conclusion, although many of the details are yet to be fully explored, the formula we have proven above for the first time allows us to consider many different results relating rank one conditions and tau-functions in a unified context. We

plan to address questions of the relationship between the geometry of the space of matrices we utilize and the geometry of the grassmannian, reductions to finite dimensional Hamiltonian (particle) systems, and the case in which A, B and C are taken to be infinite dimensional operators in a future paper.

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